Dissipation statistics of a passive scalar in a multidimensional smooth flow *

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Abstract

We compute analytically the probability distribution function $\mathcal{P}(\varepsilon)$ of the dissipation field $\varepsilon = (\nabla\theta)^2$ of a passive scalar θ advected by a d-dimensional random flow, in the limit of large Peclet and Prandtl numbers (Batchelor-Kraichnan regime). The tail of the distribution is a stretched exponential: for $\varepsilon \to \infty$, $\ln \mathcal{P}(\varepsilon) \sim -(d^2\varepsilon)^{1/3}$.

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1 Introduction

Intermittency and strong non-gaussianity of developed turbulence are most clearly reflected in the peculiar structure of the observed probability distribution functions (p.d.f.) of the gradients of the turbulent field. A typical

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logarithmic plot of the gradients p.d.f. is concave rather than convex, showing a strong central peak and slowly decaying tails. Rare strong fluctuations are responsible for the tails, while large quiet regions are related to the central peak. In particular, such p.d.f. were observed for the square gradients $\varepsilon = (\nabla\theta)^2$ (dissipation field) of a scalar field θ passively advected by an incompressible turbulent flow.

From the theoretical standpoint, most rigorous results in the theory of turbulent mixing have been obtained so far within the framework of the Kraichnan model [1, 2], describing the advection of a passive scalar by a random velocity field, delta-correlated in time. The exact solvability of the Kraichnan model suggests that it can play in turbulence a role similar to that played by the Ising model in the theory of critical phenomena [1].

In particular, the p.d.f. of the dissipation field ε was recently calculated analytically in the one-dimensional [3] and two-dimensional [4] cases. In this paper we extend the result of Ref. [4] to arbitrary space dimensions d. Our techniques are based on a combination of functional integration and group-theoretical methods, first introduced by one of us [5] in the context of quantum magnetism. These techniques allowed first to compute exactly the p.d.f. of the passive scalar in the two-dimensional, non-dissipative case [6]. The multidimensional case was considered in [8, 9, 10]. In the dissipative case, an essential ingredient is the time-separation method introduced in [3] in a one-dimensional context, and afterwards exploited in Ref. [4]. The point is that in developed turbulence there appears a natural small parameter, $1/Pe$, where Pe is the Peclet number, measuring the relative strength of advection with respect to diffusion. One would like to use this parameter to develop an asymptotic theory for quantities like the p.d.f. $\mathcal{P}(\varepsilon)$ of the dissipation field ε. However, such quantities are essentially non perturbative in $1/Pe$. For instance, ε is exactly zero without diffusion, but it has a non-zero limit as $1/Pe \neq 0$, $Pe \to \infty$. [2]

[1] The similarity should be understood in a loose sense. The Kraichnan model can be viewed in itself as a sort of mean-field approximation to a full theory of developed turbulence [7].

[2] It is worth noting that without dissipation stationary limits for the p.d.f.'s of the passive scalar field, of its gradients, *etc.*, do not exist. In the case of the direct cascade considered in this paper an arbitrary small but finite dissipation constant κ provides stationary distributions at large times for all the physical quantities. However, there are cases corresponding to the dynamical inverse cascade when dissipation does not lead to

Following Refs. [3, 4], we show that an appropriate dynamical formalism naturally introduces two time-scales, a short one related to stretching, and a long one related to diffusion. The time scale of the stretching fluctuations is of the order of the inverse of the maximum Lyapunov exponent, while the whole time of stretching is $\ln Pe$ times larger.

The paper is organised as follows:

In Sec. 2 we define the problem, recall the statistical and kinematical concepts involved, introduce the functional transformation which makes the problem solvable and write down the corresponding functional Jacobian.

In Sec. 3 we expose the time-separation method, showing that the computation of $\mathcal{P}(\varepsilon)$ can be split in two parts, which we will denote as small- and large-times averaging. Suggestively, this procedure can be thought of as a two-step application of a renormalisation group transformation.

In Secs. 4 and 5 we perform respectively large- and small-times averaging, reducing the computation of the corresponding functional integrals to solvable auxiliary quantum-mechanical problems.

In Sec. 6 we put together all the pieces of the computation and present the resulting p.d.f. $\mathcal{P}(\varepsilon)$, whose asymptotic behaviour is discussed as a function of the space dimension d.

2 Statement of the problem

The advection of a passive scalar field $\theta(t, \mathbf{r})$ by the incompressible smooth flow $\mathbf{v}(t, \mathbf{r})$ in d-dimensional space is governed by the transport equation

$$(\partial_t + \mathbf{v} \cdot \nabla - \kappa \triangle)\, \theta = \phi \,, \tag{1}$$

where $\phi(t, \mathbf{r})$ is an external source, κ is the diffusivity and we assume a null initial condition at $t = -\infty$. We assume that both $\mathbf{v}(t, \mathbf{r})$ and $\phi(t, \mathbf{r})$ are Gaussian independent random functions, δ-correlated in time [13, 2, 14, 6] so that, firstly:

$$\langle \phi(t_1, \mathbf{r}_1) \phi(t_2, \mathbf{r}_2) \rangle = \delta(t_1 - t_2) \chi(r_{12}) \,, \tag{2}$$

where $r_{12} \equiv |\mathbf{r}_1 - \mathbf{r}_2|$, the correlation $\chi(r_{12})$ decays on the scale L, and the constant $P_2 \equiv \chi(0)$ is the production rate of θ^2. We consider large Prandtl

stationary p.d.f.'s for some observables [11].

numbers, which correspond to a large viscosity-to-diffusivity ratio, so that in the viscous interval

$$\langle v_\alpha(t_1,\mathbf{r}_1)v_\beta(t_2,\mathbf{r}_2)\rangle = \delta(t_1-t_2)\Big[V_0\delta_{\alpha\beta} - \mathfrak{D}(r^2\delta_{\alpha\beta} - r_\alpha r_\beta) - \frac{(d-1)\mathfrak{D}}{2}\delta_{\alpha\beta}r^2\Big]\,. \tag{3}$$

Isotropy of the velocity statistics is here assumed. The representation (3) is valid for scales smaller than the velocity infrared cut-off L_u (which is supposed to be the largest scale of the problem), since (3) represents the two first terms of the expansion of the velocity correlation function in powers of r/L_u, and $\mathfrak{D} \sim V_0/L_\mathrm{u}^2$. We assume also that the inequality $Pe^2 \equiv \mathfrak{D}L^2/2\kappa \gg 1$ holds, guaranteeing the existence of a convective interval of scales $r_\mathrm{diff} \ll r \ll L$ where the dominant effect to be taken into account is the stretching of the velocity field. Here $r_\mathrm{diff} = 2\sqrt{\kappa/(d-1)\mathfrak{D}}$ is the mean diffusion length.

It follows from (3) that the correlation functions of the traceless strain field, defined as $\sigma_{\alpha\beta} = \nabla_\beta v_\alpha$, are $\mathbf{r}$-independent:

$$\langle\sigma_{\alpha\beta}(t_1)\sigma_{\mu\nu}(t_2)\rangle = \mathfrak{D}\left[(d+1)\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} - \delta_{\alpha\beta}\delta_{\mu\nu}\right]\delta(t_1-t_2)\,. \tag{4}$$

This means that the strain field $\sigma_{\alpha\beta}$ can be treated as a random function of time t only. To exploit this property it is convenient to pass to a comoving reference frame, that is, to a coordinate frame whose origin follows the motion of a Lagrangian particle of the fluid, so that the velocity field can be approximated by $\mathbf{v} \simeq \hat\sigma(t)\mathbf{r}$, with $\langle\hat\sigma_{\alpha\beta}(t)\rangle = 0$ [6]. With this substitution, (1) can be solved by Fourier transforming in the space coordinates and integrating along characteristics:

$$\theta_\mathbf{k}(t) = \int_{-\infty}^{t}\mathrm{d}t'\,\phi(t',\hat{\mathcal{W}}^\mathrm{T}(t,t')\mathbf{k})\exp\Big[-\kappa\,\mathbf{k}\cdot\int_{t'}^{t}\mathrm{d}t_1\hat{\mathcal{W}}(t,t_1)\hat{\mathcal{W}}^\mathrm{T}(t,t_1)\mathbf{k}\Big] \tag{5}$$

where

$$\hat{\mathcal{W}}(t,t') = \mathcal{T}\exp\left(\int_{t'}^{t}\hat\sigma(\tau)\mathrm{d}\tau\right) \tag{6}$$

is the solution of the evolutive problem

$$\dot{\hat{\mathcal{W}}}(t) = \hat\sigma(t)\hat{\mathcal{W}}(t), \qquad \hat{\mathcal{W}}(t') = \mathbf{1}\,, \tag{7}$$

and $\mathcal{T}$ is the chronological ordering operator. We will study stationary statistical properties of the passive scalar at the fixed time moment t. It is

convenient to use the symmetries of the measure of averaging over $\hat{\sigma}(\tau)$ with respect to the transformations $\hat{\sigma}(\tau) \longrightarrow \hat{\sigma}(-\tau)$ and $\hat{\sigma}(\tau) \to -\hat{\sigma}(\tau)$. Changing now variables in the time integral in (5) we arrive to the expression:

$$\theta_{\mathbf{k}}(t) = \int_0^\infty \mathrm{d}t' \, \phi(t-t', \hat{\mathcal{W}}^{-1,\mathrm{T}}(t')\mathbf{k}) \exp\left[-\kappa \, \mathbf{k} \int_0^{t'} \mathrm{d}\tau \hat{\mathcal{W}}^{-1}(\tau)\hat{\mathcal{W}}^{-1,\mathrm{T}}(\tau)\mathbf{k}\right]. \quad (8)$$

Here $\hat{\mathcal{W}}(t) \equiv \hat{\mathcal{W}}(t,0)$ and invariance of the random force ϕ averaging with respect to time inversion was used. Let us now consider the statistics of the dissipation field

$$\varepsilon = \kappa(\nabla\theta)^2 .$$

The p.d.f. of the ε field can be written as

$$\mathcal{P}(\varepsilon) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}s \, e^{s\varepsilon} \int \frac{\mathrm{d}^d m}{\pi^{d/2}} \, e^{-m^2} \, \langle e^{-sQ} \rangle_{\hat{\sigma}} \, , \quad (9)$$

with

$$Q = \frac{\mathfrak{D}}{Pe^2} \int_0^\infty \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \chi_k \, \left(\mathbf{k}\cdot\hat{\mathcal{W}}(t)\mathbf{m}\right)^2 \, \exp\left(-\frac{\mathfrak{D}}{Pe^2} \, \mathbf{k}\cdot\hat{\Lambda}(t)\mathbf{k}\right) \quad (10)$$

and

$$\hat{\Lambda}(t) = \hat{\mathcal{W}}(t) \left(\int_0^t \hat{\mathcal{W}}^{-1}(\tau)\hat{\mathcal{W}}^{-1,\mathrm{T}}(\tau)\mathrm{d}\tau \right) \hat{\mathcal{W}}^{\mathrm{T}}(t) \quad (11)$$

Here the integration over the auxiliary vector parameter $\mathbf{m}$ takes care of combinatorics and summation over vector indices. The moments of the dissipation field ε can be recovered as

$$\langle \varepsilon^n \rangle = \int \frac{\mathrm{d}^d m}{\pi^{d/2}} \, e^{-m^2} \, \langle Q^n \rangle_{\hat{\sigma}} \, . \quad (12)$$

The average $\langle \ldots \rangle_{\hat{\sigma}}$ over the traceless random strain field $\hat{\sigma}$ has to be performed with the probability measure $\mathcal{D}\hat{\sigma}(t) \exp(-S_{\hat{\sigma}})$ corresponding to (4):

$$S_{\hat{\sigma}} = \frac{1}{2d(d+2)\mathfrak{D}} \int_0^{+\infty} [(d+1)\mathrm{Tr}(\hat{\sigma}\hat{\sigma}^{\mathrm{T}}) + \mathrm{Tr}(\hat{\sigma}^2)]\mathrm{d}t \, . \quad (13)$$

The matrix $\hat{\Lambda}(t)$ is invariant under right local (time dependent) rotations $\hat{\mathcal{W}}(t) \to \hat{\mathcal{W}}(t)\hat{\mathcal{R}}(t)$, and transforms covariantly under left local rotations

$\hat{\mathcal{W}}(t) \to \hat{\mathcal{R}}(t)\hat{\mathcal{W}}(t)$. The quantity Q is invariant under left local rotations, which for any fixed t can be absorbed in the $\mathrm{d}^d k$ integration. The action $S_{\hat{\sigma}}$ is invariant under both left and right global rotations and under the transformation $\hat{\sigma} \to \hat{\sigma}^{\mathrm{T}}$. Together with isotropy of the pumping function χ, invariance of $S_{\hat{\sigma}}$ and $\hat{\Lambda}$ under right rotations implies that $Q \equiv Q(\mathbf{m})$ is a function of the square modulus $z = \mathbf{m}^2$, so that we can substitute $\mathbf{m} \to \sqrt{z}\mathbf{n}_0$ in (10), with

$$\mathbf{n}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{14}$$

The m-integration measure in (9) and (12) can then be substituted by

$$\frac{1}{\Gamma(d/2)} z^{\frac{d}{2}-1} e^{-z}\, \mathrm{d}z \tag{15}$$

After these substitutions Q maintains invariance under left local rotations of $\hat{\mathcal{W}}$.

Let us perform the Iwasawa decomposition of the evolution matrix $\hat{\mathcal{W}}$:

$$\hat{\mathcal{W}}(t) = \hat{\mathrm{R}}\hat{\mathrm{D}}\hat{\mathrm{T}}^{-1}\,, \tag{16}$$

where, for any fixed t, $\hat{\mathrm{R}}$ is an $\mathrm{SO}(d)$ rotation matrix, $\hat{\mathrm{D}}$ a diagonal matrix with $\det \hat{\mathrm{D}} = 1$, and $\hat{\mathrm{T}}$ an upper triangular matrix with 1's on the diagonal.

The action $S_{\hat{\sigma}}$ takes then the form:

$$\begin{aligned} S_{\hat{\sigma}} \quad = \quad & \frac{1}{2d\mathfrak{D}} \int_0^\infty \mathrm{d}t \left\{ \mathrm{Tr}(\dot{\hat{\mathrm{D}}}\hat{\mathrm{D}}^{-1})^2 + \frac{1}{2}\mathrm{Tr}[\hat{\mathrm{D}}^2(\hat{\mathrm{T}}^{-1}\dot{\hat{\mathrm{T}}})\hat{\mathrm{D}}^{-2}(\hat{\mathrm{T}}^{-1}\dot{\hat{\mathrm{T}}})^{\mathrm{T}}] \right. \\ & \left. - \frac{d}{d+2}\mathrm{Tr}[\hat{\mathrm{R}}^{-1}\dot{\hat{\mathrm{R}}} - (\hat{\mathrm{D}}\hat{\mathrm{T}}^{-1}\dot{\hat{\mathrm{T}}}\hat{\mathrm{D}}^{-1})_{\mathrm{a}}]^2 \right\} . \end{aligned} \tag{17}$$

where $(\ldots)_{\mathrm{a}}$ denotes the antisymmetric part.

The invariance of Q under local rotations of $\hat{\mathcal{W}}$ implies that Q does not depend on $\hat{\mathrm{R}}$, so that the $\hat{\mathrm{R}}$ variables can be integrated out from the very beginning. The resulting effective action is equal to $S_{\hat{\sigma}}$ with the last trace-term dropped. It is then convenient to parametrize

$$\hat{\mathrm{D}} = \begin{pmatrix} e^{\rho_1}, & 0, & \ldots, & 0 \\ 0, & e^{\rho_1}, & \ldots, & 0 \\ \ldots, & \ldots, & \ldots, & \ldots \\ 0, & 0, & \ldots, & e^{-(\rho_1+\cdots+\rho_{d-1})} \end{pmatrix} \tag{18}$$

and

$$\hat{\mathrm{T}} = \begin{pmatrix} 1, & \eta_1, & \eta_2, & \dots, & \eta_{d-1} \\ 0, & 1, & X_{23}, & \dots, & X_{2,d} \\ \dots, & \dots, & \dots, & \dots, & \dots \\ 0, & 0, & 0, & \dots, & 1 \end{pmatrix}, \tag{19}$$

denoting by $\eta_j \equiv X_{1,j+1}$ the elements of the first row of $\hat{\mathrm{T}}$, which, together with the ρ_j, will be seen to be the only relevant dynamical variables. In order to have more compact expressions it will be also convenient in what follows to denote the sum $\rho_1 + \dots + \rho_{d-1}$ by $-\rho_d$ and to set $X_{jj} \equiv 1$.

The initial condition $\hat{\mathcal{W}}(0) = \mathbf{1}$ implies

$$\begin{aligned} \rho_j(0) &= 0, \quad \eta_j(0) = 0, \quad j = 1, \dots, d-1\,, && (20) \\ X_{mn}(0) &= 0, \quad 2 \le m < n \le d\,. && (21) \end{aligned}$$

With the substitution $\hat{\sigma} \to (\hat{\mathrm{R}}, \hat{\mathrm{D}}, \hat{\mathrm{T}})$ we introduced in the functional integral a set of new "collective" coordinates, by means of a non-linear, non-local variable transformation. The Jacobian of the transformation is given by

$$\mathcal{J} = \mathcal{J}_{\mathrm{ul}} \cdot \mathcal{J}_{\mathrm{l}} = \prod_{t=0}^{\infty} \exp\left[2 \sum_{j=1}^{d-1} (d-j)\rho_j(t)\right] \cdot \exp\left[\int_0^\infty \sum_{j=1}^{d-1} (d-j)\dot{\rho}_j(\tau) \mathrm{d}\tau\right]. \tag{22}$$

The details of the derivation are given in the Appendix. The local part $\mathcal{J}_{\mathrm{l}}$ of the Jacobian coincides with the one computed in Ref. [8].

With the decomposition (16), and after the integration of the $\hat{\mathrm{R}}$ variables, the probability measure takes the form

$$\prod_{j=1}^{d-1} \mathcal{D}\rho_j \mathcal{D}\eta_j \prod_{2 \le m < n \le d} \mathcal{D}X_{mn}\, \mathcal{J}(\rho)\, \exp\left(-S_{\mathrm{eff}}[\rho, \eta, X]\right),$$

with

$$S_{\mathrm{eff}} = \frac{1}{2d\mathfrak{D}} \int_0^\infty \mathrm{d}t \Big[\sum_{j=1}^{d} \dot{\rho}_j^2 + \frac{1}{2} \sum_{1 \le i < j \le d} e^{2(\rho_i - \rho_j)} \xi_{ij}^2\Big] \tag{23}$$

and $\hat{\xi} = \hat{\mathrm{T}}^{-1} \dot{\hat{\mathrm{T}}}$. Let us also rewrite (10) as

$$Q = \frac{z\mathfrak{D}}{Pe^2} \int_0^\infty \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \chi_k\, k_1^2\, e^{2\rho_1}\, \exp\left(-\frac{\mathfrak{D}}{Pe^2}\, \mathbf{k} \cdot \hat{\Lambda}(t) \mathbf{k}\right), \tag{24}$$

whereby fixing the residual left-rotation symmetry.

3 Time separation

We shall now expose a method that allows to compute $\mathcal{P}(\varepsilon)$ in the limit of large Peclet numbers, *i.e.*, in a regime characterized by a large ratio of advection to diffusion at the pumping scale.

The moments of the dissipation field ε have finite, non zero limits as diffusivity tends to zero. Thus, $\mathcal{P}(\varepsilon)$ is a non-perturbative quantity in the small parameter $1/Pe$.

The time separation method, which was introduced in Ref. [3] in a one-dimensional context and generalized to $d = 2$ in [4], is the proper tool for performing such a non-perturbative calculation. As a matter of fact, the dynamical formalism itself reveals the presence of two different time scales: a short one, related to diffusion, and a long one, related to advective stretching. Taking properly into account the two time scales it is possible to exploit non-trivially the presence of the large parameter Pe and to develop an asymptotic theory that captures the dominant term in $\mathcal{P}(\varepsilon)$ with respect to Pe.

The starting point is that, as is clear from the structure of (24), the main contribution to the moments Q^n is obtained in the limit $Pe \to \infty$ for $\rho_1 \sim \ln Pe$. On the other hand, one has the following result (see [9]) for the hierarchy of the Lyapunov exponents:

$$\langle \rho_j(t) \rangle_{\hat{\sigma}} = \frac{d}{2}\mathfrak{D}(d-2j+1)t \equiv \bar{\lambda}_j t \,, \tag{25}$$

showing that the relevant values of ρ_1 are realized for times

$$t \simeq \frac{2}{d(d-1)\mathfrak{D}} \ln Pe \,. \tag{26}$$

For large times the elements of the matrix $\Lambda(t)$ can be estimated roughly as

$$\Lambda_{jl}(t) \sim \delta_{jl} e^{2\rho_j(t)}, \quad t \to \infty$$

and from (25) it follows that at times of the order (26) the components $\Lambda_{jl}(t)$ with $j, l \neq 1$ are small compared with Pe^2 and can be neglected in the diffusive exponent (24). Thus, only Λ_{11} effectively survives.

Let us then introduce a separation time t_0 such that $1 \ll d^2\mathfrak{D}\, t_0 \ll \ln Pe$. In the limit $Pe \to \infty$ this separation time will disappear from the final answer. The contribution to Q from the interval $0 < t < t_0$ is parametrically small, so that we can substitute $0 \to t_0$ as a lower integration limit in (24).

Observe now that for the trajectories contributing to $\langle Q^n \rangle$ the ξ_{ij} fields, being multiplied in the action (23) by a factor $\exp(2\rho_i - 2\rho_j) \gg 1$, will be exponentially depressed for $t \gg 1$. The components of $\hat{\mathrm{T}}$ will therefore be frozen to constant values for those times. This is in particular true for the components η_j of the first row.

Taking into account the freezing of $\hat{\mathrm{T}}$ and writing explicitly $\hat{\Lambda}$ in the parametrization (18,19) it is seen that the integral in (11) saturates for $t \simeq t_0$, so that the *upper* integration limit in (11) can be substituted with t_0. As a final result, in the limit $Pe \to \infty$, (24) asymptotically reduces to

$$Q = \frac{z\mathfrak{D}}{Pe^2} \int_{t_0}^{\infty} \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d} \chi_k \, k_1^2 \, e^{2\rho_1(t)} \times \tag{27}$$

$$\times \exp\left[-\frac{\mathfrak{D}}{Pe^2} k_1^2 e^{2\rho_1(t)} \, \mathbf{n}_0 \cdot \int_0^{t_0} \mathrm{d}\tau \hat{\mathrm{T}}^{-1}(t_0) \hat{\mathcal{W}}^{-1}(\tau) \hat{\mathcal{W}}^{-1,\mathrm{T}}(\tau) \hat{\mathrm{T}}^{\mathrm{T}}(t_0) \mathbf{n}_0 \right]$$

Observe that all the dependence of Q on the stochastic variables labelled by small times $t < t_0$ is now concentrated in the square-bracketed exponent of (27). The action (23) can also be written as a sum $S_{\mathrm{eff}}^{<} + S_{\mathrm{eff}}^{>}$ of a small- and a large-times part. This makes clear the statement that integration over small-times variables, which describe a mainly dissipative process, can be separated from integration over large-times ($t > t_0$) variables, which describe multiplicative stretching. Time separation appears therefore as a natural consequence of the dynamic formalism.

The small-times and large-times variables are coupled only by the condition of continuity at $t = t_0$:

$$\rho_j^{>}(t_0) = \rho_j^{<}(t_0), \quad \eta_j^{>}(t_0) = \eta_j^{<}(t_0), \quad X_{mn}^{>}(t_0) = X_{mn}^{<}(t_0) \,. \tag{28}$$

In order to fully implement time separation we perform the reparametrization

$$\hat{\mathcal{W}}(t) \to \hat{\mathcal{W}}(t) \hat{\mathrm{T}}^{-1}(t_0) \,.$$

which implies the substitution $\hat{\mathrm{T}}(t_0) \to \mathbf{1}$ in (27) and modifies the boundary conditions (20,21) and (28) as follows:

$$\rho_j^{<}(0) = 0, \quad \eta_j^{<}(t_0) = X_{mn}^{<}(t_0) = 0 \,, \tag{29}$$

for the small-times fields, and

$$\rho_j^{>}(t_0) = \rho_j^{<}(t_0), \quad \eta_j^{>}(t_0) = X_{mn}^{>}(t_0) = 0 \tag{30}$$

for the large times.

Thus, all the dissipative effects are encoded in the p.d.f. of the only variable

$$\mu = \mathbf{n}_0 \cdot \int_0^{t_0} \mathrm{d}\tau\, \hat{\mathcal{W}}^{-1}(\tau)\hat{\mathcal{W}}^{-1,\mathrm{T}}(\tau)\mathbf{n}_0\,, \tag{31}$$

which appears as a parameter in the large-times averaging. In order to compute the total generating function

$$\mathcal{P}_{s,z} = \langle e^{-sQ} \rangle \tag{32}$$

we shall integrate *first* on the large-times variables (essentially, only ρ_1), find a p.d.f. $\mathcal{P}^{>}_{s,z}$ which depends on μ as a parameter, and then complete the computation by integrating over the small-times p.d.f. $\mathcal{P}^{<}(\mu)$ This procedure can be suggestively thought as a renormalization group transformation consisting of only two steps.

Let us now outline the first step. First of all, the variables $\eta^{>}_j$, $X^{>}_{mn}$ can be integrated out completely, since they do not appear in (27). Of the $\rho^{>}_j$ variables, the only $\rho^{>}_1$ appears in (27). Integration over $\rho^{>}_1$ can be separated from the trivial integration over $\rho^{>}_2, \ldots, \rho^{>}_{d-1}$ by the simple shift

$$\rho^{>}_1(t) = \tilde{\rho}_1(t) + \rho_1(t_0), \quad \rho^{>}_j(t) = \tilde{\rho}_j(t_0) - \frac{\tilde{\rho}_1(t)}{d-1} + \rho_j(t_0),\ j = 2, \ldots, d-1\,, \tag{33}$$

which also substitutes the initial condition $\rho^{>}_j(t_0) = \rho^{<}_j(t_0)$ with the simpler

$$\tilde{\rho}_j(t_0) = 0, \quad j = 1, \ldots, d-1\,. \tag{34}$$

Finally, we are left with an integration over the effective measure already obtained in Ref. [8]:

$$\mathcal{D}\rho_1\ \exp\left[-\frac{1}{2(d-1)\mathfrak{D}} \int_{t_0}^{\infty} \mathrm{d}t\ \left(\dot{\tilde{\rho}}_1 - \frac{d(d-1)\mathfrak{D}}{2}\right)^2\right] \tag{35}$$

and with the factor

$$\mathcal{J}_0(\rho) = \exp\left[\sum_{j=1}^{d-1}(d-j)\rho_j(t_0)\right]\,, \tag{36}$$

which will be part of the small-times averaging measure.

4 Large-times averaging

Let us now apply the previous considerations to the actual evaluation of $\mathcal{P}^{>}_{s,z}$, defined as the average of e^{-sQ} with respect to the measure (35) with the constraint $\tilde{\rho}_1(t_0) = 0$. The evaluation of the corresponding path integral can be reduced via the Feynman-Kac formula to the solution of the following auxiliary quantum mechanical problem in the space of the variable $\rho \equiv \tilde{\rho}_1$:

$$\begin{aligned}\mathcal{P}^{>}_{s,z} &= \lim_{T\to\infty} e^{-\frac{d^2(d-1)\mathfrak{D}}{8}(T-t_0)} \langle \delta(\rho) | \exp[-(T-t_0)\hat{\mathcal{H}}^{>}] | e^{\frac{d}{2}\rho} \rangle \\ &= \lim_{T\to\infty} e^{-\frac{d^2(d-1)\mathfrak{D}}{8}(T-t_0)} \Psi(T-t_0; \rho = 0) , \end{aligned} \tag{37}$$

with

$$\hat{\mathcal{H}}^{>} = -\frac{(d-1)\mathfrak{D}}{2} \frac{\partial^2}{\partial \rho^2} + V^{>}(\rho) , \tag{38}$$

$$V^{>}(\rho) = sz\beta \frac{\mathfrak{D}}{Pe^2} \cdot e^{2\rho} \cdot \Xi\left(\sqrt{\frac{\mathfrak{D}}{Pe^2}\beta\mu}\, e^{\rho}\right) , \quad \beta = e^{2\rho(t_0)} , \tag{39}$$

$$\Xi(y) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \chi_k\, k_1^2 e^{-k_1^2 y^2} . \tag{40}$$

In (37), the "wave function" Ψ is the solution of the initial value problem

$$\hat{\mathcal{H}}^{>}\Psi = -\partial_t \Psi , \tag{41}$$

$$\Psi(t=0;\rho) = e^{\frac{d}{2}\rho} . \tag{42}$$

For $t \to \infty$ the behaviour of $\Psi(t,\rho)$ is dominated by the lowest eigenvalue of the corresponding stationary problem, which can be easily determined since the potential $V^{>}(\rho)$ vanishes at $\rho \to \infty$, leaving us with a free Hamiltonian:

$$\hat{\mathcal{H}}^{>} e^{\frac{d}{2}\rho} \simeq -\frac{(d-1)\mathfrak{D}}{2}\frac{d^2}{4} e^{\frac{d}{2}\rho}, \quad \rho \to \infty .$$

So, for $t \to \infty$:

$$\Psi(t;\rho) \simeq e^{\frac{d^2(d-1)\mathfrak{D}}{8}t} e^{\frac{d}{2}\rho} \Phi(y) , \tag{43}$$

with $\Phi(y)$ the solution of the following boundary problem in the variable $y = \sqrt{\mathfrak{D}\beta\mu/Pe^2}\, e^{\rho}$:

$$\left[-\frac{1}{y^{d+1}} \frac{\partial}{\partial y} y^{d+1} \frac{\partial}{\partial y} + \frac{2sz}{(d-1)\mathfrak{D}\mu} \Xi(y) \right] \Phi(y) \;=\; 0\,, \tag{44}$$

$$\Phi(y) \to 1, \quad y \to \infty; \qquad y\Phi(y) \to 0, \quad y \to 0\,. \tag{45}$$

Note that $\Phi(y)$ can depend on the parameters s, z, $\mathfrak{D}$ and μ only through their combination

$$\lambda = \frac{2sz}{(d-1)\mathfrak{D}\mu}\,, \tag{46}$$

and does not depend on Pe.

Finally, from (37,43) it follows

$$\mathcal{P}^{>}_{s,z} = \Phi\left(\sqrt{\frac{\mathfrak{D}}{Pe^2}\beta\mu} \right)\,. \tag{47}$$

Here $\mathcal{P}^{>}_{s,z}$ depends on Pe only through the argument of Φ. In the limit $Pe \to \infty$ one simply finds

$$\mathcal{P}^{>}_{s,z} = \Phi(0) \tag{48}$$

For a generic given χ_k, one can solve (44) iteratively in terms of the small potential $s\Xi(y)$, $s \to 0$, finding explicit expressions for all the "large-times" moments of Q:

$$\begin{aligned} \mathcal{P}^{>}_{s,z} \;=\; & 1 + \sum_{k=1}^{\infty} (2k-1)!! \cdot \left(\frac{2s\beta}{(d-1)\mathfrak{D}\mu} \right)^k \times \\ & \times \int_{\substack{y_{2k} < y_{2k-1} \\ y_{2k} < y_{2k+1}}} \mathrm{d}y_1 \cdots \mathrm{d}y_{2k} \prod_{l=1}^{k} \left(\frac{y_{2l}}{y_{2l-1}} \right)^{d+1} \Xi(y_{2l}) \end{aligned} \tag{49}$$

The moments are here expressed directly in terms of the integral transform (40) of the pumping function χ_k.

5 Small times averaging

In this section we shall compute

$$\mathcal{P}^{<}_{s} = \langle e^{-s\mu} \rangle\,. \tag{50}$$

The explicit expression of μ is

$$\mu = \int_0^{t_0} V^{<} \mathrm{d}t$$

with

$$V \equiv V^{<} = e^{-2\rho_1} + \eta_1^2 e^{-2\rho_2} + \eta_2^2 e^{-2\rho_3} \cdots + \eta_{d-1}^2 e^{-2\rho_d} \tag{51}$$

In what follows we will omit the $(\ldots)^{<}$ index on the ρ_j and η_j variables. Along the lines previously exposed in the case of large-times averaging, the computation of $\mathcal{P}_s^{<}$ is again reduced to the solution of an auxiliary quantum mechanical problem:

$$\mathcal{P}_s^{<} = \exp\left(-\frac{d^2(d^2-1)\mathfrak{D}}{24}\, t_0 \right) \times \tag{52}$$
$$\times \Big\langle \prod_{j=1}^{d-1} e^{(d-j)\rho_j}\, \delta(\eta_j) \prod_{2\le m<n\le d} \delta(X_{mn}) \Big| \exp\left(-t_0 \hat{\mathcal{H}}^{<} \right) \Big| \prod_{j=1}^{d-1} \delta(\rho_j) \Big\rangle\,.$$

Here

$$\hat{\mathcal{H}}^{<} = \hat{\mathcal{H}}_0 + sV\,,$$

and $\hat{\mathcal{H}}_0$ is the quantum Hamiltonian corresponding to the classical action (23):

$$\hat{\mathcal{H}}_0 \;=\; -\frac{(d-1)\mathfrak{D}}{2} \sum_{j=1}^{d-1} \frac{\partial^2}{\partial {\rho_j}^2} + \mathfrak{D} \sum_{0<i<j<d} \frac{\partial^2}{\partial\rho_i \partial\rho_j} \tag{53}$$
$$-d\mathfrak{D} \sum_{1\le j<i\le d} e^{2\rho_i - 2\rho_j} \Big(\sum_{k\le j;\; k<i} X_{kj} \frac{\partial}{\partial X_{ki}} \Big)^2 \tag{54}$$

Due to the the triangular structure of the X_{mn} and η_j variables there arise no ordering problems.

The seemingly intractable quantum mechanical problem in $\frac{1}{2}d(d+1)-1$ dimensions can in fact be reduced to a solvable, one-dimensional problem thanks to the presence of $\frac{1}{2}d(d+1)-2$ symmetries. First of all, the initial wave function in (52) does not depend on the X_{jl} variables. The evolution operator $\exp(-t_0\hat{\mathcal{H}}^{<})$ introduces a dependence on the η_j variables, but not on the remaining X_{ij} $(i>1)$, as is clear from the explicit expressions (51)

and (54). We are thus led to consider the reduction $\hat{\mathcal{H}}^{(\rho,\eta)}$ of $\hat{\mathcal{H}}^{<}$ on the space of functions depending only on ρ_j and η_j:

$$\hat{\mathcal{H}}^{(\rho,\eta)} = -\frac{(d-1)\mathfrak{D}}{2}\sum_{j=1}^{d-1}\frac{\partial^2}{\partial\rho_j^2} + \mathfrak{D}\sum_{0<i<j<d}\frac{\partial^2}{\partial\rho_i\partial\rho_j} \tag{55}$$

$$-d\mathfrak{D}\sum_{j=2}^{d}\left(e^{-2(\rho_1-\rho_j)} + \sum_{k=2}^{j-1} e^{-2(\rho_k-\rho_j)}\eta_{k-1}^2\right)\frac{\partial^2}{\partial\eta_{j-1}^2} + V\,. \tag{56}$$

We can then forget altogether about the X_{jl} variables and just compute

$$\left\langle e^{\sum_{j=1}^{d-1}(d-j)\rho_j}\delta(\eta_1)\cdots\delta(\eta_{d-1})\middle|\exp\left(-t_0\hat{\mathcal{H}}^{(\rho,\eta)}\right)\middle|\delta(\rho_1)\cdots\delta(\rho_{d-1})\right\rangle. \tag{57}$$

The reduced Hamiltonian $\hat{\mathcal{H}}^{(\rho,\eta)}$ depends on $2d-2$ variables. Let us use now the fact that the action $S^{<} = S^{<}_{\text{eff}} + s\mu$ is invariant under the global left transformations

$$\hat{\mathrm{T}}(t) \to \hat{\Theta}\hat{\mathrm{T}}(t)\,, \tag{58}$$

where we take $\hat{\Theta} = \mathbf{1} + \delta\hat{\Theta}$, with $\delta\hat{\Theta}$ having non zero values only in the first column and on the diagonal, and satisfying $\mathrm{Tr}(\delta\hat{\Theta}) = 0$ and $(\delta\hat{\Theta})_{00} = 0$. This way we generate $2d-3$ symmetries of $S^{<}$, which are readily identified with rotations in the space of the variables $\eta_{j-1}e^{2\rho_j}$ and shifts of the ρ_j accompanied by corresponding rescalings of the η_j. The corresponding variations of $\hat{\mathrm{D}}$ and $\hat{\mathrm{T}}$ can be read out from

$$\left(\hat{\mathrm{D}}\hat{\mathrm{T}}^{-1}\delta\hat{\Theta}\hat{\mathrm{T}}\hat{\mathrm{D}}^{-1}\right)_{\mathrm{s}} = \left(\delta\hat{\mathrm{D}}\hat{\mathrm{D}}^{-1} - \hat{\mathrm{D}}\hat{\mathrm{T}}^{-1}\delta\hat{\mathrm{T}}\hat{\mathrm{D}}^{-1}\right)_{\mathrm{s}} \tag{59}$$

where $(\ldots)_{\mathrm{s}}$ denotes the symmetric part. The corresponding integrals of the motion are also easily found. They locally generate $2d-3$ "angular" directions, transverse to the "radial" direction labelled by V. We will not need their explicit expression: it will be enough to determine the "radial" part of $\hat{\mathcal{H}}^{(\rho,\eta)}$ by the transformation $\partial_x = \partial V/\partial x \cdot \partial_V$ +(radial part), with x substituted by ρ_j, η_j. In the $t_0 \gg 1$ limit (57) can be computed explicitly as

$$\mathcal{P}_s^{<} = \int_{-\infty}^{\infty}\mathrm{d}\eta_1\cdots\int_{-\infty}^{\infty}\mathrm{d}\eta_{d-1}\,\Phi(\rho_j = 0, \eta_j)\,, \tag{60}$$

with Φ the solution of the stationary problem

$$\hat{\mathcal{H}}^{(\rho,\eta)}\Phi(\rho,\eta) \;=\; 0\,, \tag{61}$$

$$\Phi(\rho,\eta) \;\simeq\; \delta(\eta_1)\cdots\delta(\eta_{d-1})\,\exp\Big[\sum_{j=1}^{d-1}(d-j)\rho_j\Big]\,,\quad \rho\to\infty\,, \tag{62}$$

$$\Phi(\rho,\eta) \;\simeq\; 0\,,\quad \rho\to-\infty\,. \tag{63}$$

Let us consider a maximally symmetric solution Φ having the form

$$\Phi(\rho,\eta) = \exp\Big[\sum_{j=1}^{d-1}(d-j)\rho_j\Big]\mathcal{F}(V)\,, \tag{64}$$

with $\mathcal{F}(V)$ satisfying the "radial" equation

$$2(d-1)V^2\mathcal{F}'' + (d-1)(d+2)V\mathcal{F}' - \frac{sV}{\mathfrak{D}}\mathcal{F} = 0\,. \tag{65}$$

A solution of (65) is:

$$\mathcal{F}(V) = V^{-d/4}K_{d/2}\left(\sqrt{\frac{2sV}{(d-1)\mathfrak{D}}}\right)\,, \tag{66}$$

where $K_{d/2}$ is the Macdonald function of order $d/2$. The properties of the Macdonald functions [15] allow to verify directly that (64) satisfies the boundary conditions (62,63). This also shows *a posteriori* that the boundary conditions (62,63) are asymptotically invariant under the symmetries of the problem.

The integration (60) can be explicitely performed, giving

$$\mathcal{P}_s^< = \mathrm{const}\cdot\exp\left(-\sqrt{\frac{2s}{(d-1)\mathfrak{D}}}\right) \tag{67}$$

whose Laplace transform gives the p.d.f. of μ:

$$\mathcal{P}^<(\mu) = \frac{1}{\sqrt{2\pi(d-1)\mathfrak{D}}\;\mu^{3/2}}\,\exp\left(-\frac{1}{2(d-1)\mathfrak{D}\mu}\right)\,. \tag{68}$$

One can check the soundness of the whole computation by verifying on the first moment of ε that the constraint of energy conservation

$$\langle\varepsilon\rangle = \frac{1}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\chi_k$$

is satisfied.

6 Resulting distribution function

In order to restore the total p.d.f. $\mathcal{P}(\varepsilon)$ we should take the function $\mathcal{P}^{>}_{s,z}$ expressed in (48) through the solution of (44), then average over μ with the weight (68) and over z with the measure (15), and finally perform an inverse Laplace transform. As already noticed, $\Phi(y=0)$ depends on the parameters s, z, $\mathfrak{D}$ and μ only through their combination λ (see (46)):

$$\Phi(0) = f(\lambda) \tag{69}$$

The resulting $\mathcal{P}(\varepsilon)$ can be expressed via $f(\lambda)$ as:

$$\mathcal{P}(\varepsilon) = \frac{\sqrt{\mathfrak{D}}}{2\pi\Gamma(d/2)\ \sqrt{\varepsilon}} \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{0}^{+\infty} \mathrm{d}z\ f(ix)\,(-ix)^{\frac{d}{2}-1}\, z^{d-2} \exp\left(ixz^2 - \frac{\sqrt{\varepsilon\mathfrak{D}}}{z}\right) \tag{70}$$

The function $f(\lambda)$ has poles on the real negative semiaxis; the asymptotics of $\mathcal{P}(\varepsilon)$ is defined by the pole closest to the origin, which we denote as $-\lambda^*$ ($\lambda^* > 0$):

$$\mathcal{P}(\varepsilon) \sim \exp\left[-\frac{3}{2}\left(2\lambda^*\mathfrak{D}\varepsilon\right)^{1/3}\right],\ \varepsilon \to \infty\,. \tag{71}$$

The point is that when λ approaches the value $-\lambda^*$ the Hamiltonian $\hat{\mathcal{H}}^{>}$ develops a ground state energy

$$E_0 < -\frac{d^2(d-1)\mathfrak{D}}{8}$$

and $f(\lambda) = \mathcal{P}^{>}_{s,z}$ given by (37) becomes infinite. We therefore have

$$\lambda^* = \frac{d^2}{4\xi_*}$$

with ξ_* the maximum of the function $y^2\Xi(y)$ (*cfr.* (38,39)). This finally gives

$$\mathcal{P}(\varepsilon) \sim \exp\left[-\mathrm{const}\left(d^2\mathfrak{D}\varepsilon\right)^{1/3}\right],\quad \varepsilon \to \infty\,. \tag{72}$$

7 Discussion

An important point to be noticed in the previous computation, expecially with regard to the asymptotic expression (72), is that we consider in this paper *Pe* as the largest number in the theory. For any large, but finite *Pe* this means that the asymptotics (72) is valid for $\varepsilon/\varepsilon_0 \gg 1$ (where $\varepsilon_0 = \langle\varepsilon\rangle$) but smaller than $\ln Pe$. There is here a crucial difference with the computation of the p.d.f. $P(\theta)$ of the passive scalar θ itself, which was discussed in the papers [6, 8, 10, 12]. If $\theta \ll \ln Pe$ the p.d.f. $P(\theta)$ has a Gaussian form and can be obtained by taking into account only the variable ρ_1 corresponding to the highest Lyapunov exponent. However, if $\theta \gtrsim \ln Pe$, this approach does not give the correct exponent for $d > 2$, as it was observed in [10]. The correct asymptotic of $P(\theta)$ for $\theta \gg \ln Pe$ was computed in [10]; the complete p.d.f was then restored in [12], up to logarithmic corrections.

Let us estimate the domain of validity of our expressions (70,72), which were obtained in the limit $Pe \to \infty$. In the computation, when $\rho_1 \sim \ln Pe$ we neglected the dependence of Q on $\rho_2, \ldots, \rho_{d-1}$. However, due to the peculiar dependence of the potential on ρ_1, there exist trajectories for which ρ_1 remains confined in a finite domain around $\ln Pe$, while the variables ρ_j, $j = 2, \ldots, d-1$ freely evolve, eventually reaching values of the order $\ln Pe$. This happens for times of the order $t^* \sim \ln Pe$. The corresponding value of ε is $\varepsilon_{\lim} \sim \varepsilon_0 t^* \sim \varepsilon_0 \ln Pe$. For $\varepsilon \gtrsim \varepsilon_{\lim}$ the behaviour of $\mathcal{P}(\varepsilon)$ is probably modified.

The exponent of the p.d.f.'s tail (72) up to d-dependent numerical coefficient coincides with the one-dimensional results [3]. This means that the configurations of the passive scalar field contributing into (72) are effectively one-dimensional, as it was noted in [16].

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A Computation of the Jacobian

In this appendix we derive the form of the Jacobian term (22). Let us introduce a basis $\hat{\mathrm{e}}_{ij}$ of the matrix algebra $\mathrm{gl}(d)$ defined by

$$(\hat{\mathrm{e}}_{ij})_{kl} = \delta_{ik}\delta_{jl}\,,$$

make use of the notation

$$\hat{\mathrm{t}}_{ij} = \hat{\mathrm{e}}_{ij},\ j > i, \quad \hat{\mathrm{d}}_k = \hat{\mathrm{e}}_{kk},\ k = 1, \dots\, d, \quad \hat{\mathrm{r}}_{ij} = \hat{\mathrm{e}}_{ij} - \hat{\mathrm{e}}_{ji},\ j > i\,,$$

and parametrize

$$\hat{\mathrm{R}} = \mathcal{T} \exp\left(\int_0^t \phi_{ij}\hat{\mathrm{r}}_{ij}\mathrm{d}\tau\right), \quad \hat{\mathrm{D}} = \exp\left(\rho_k\hat{\mathrm{d}}_k\right), \quad \hat{\mathrm{T}}^{-1} = \exp\left(X_{ij}\hat{\mathrm{t}}_{ij}\right)\,,$$

where summation on repeated indices is assumed and $\rho_d = -(\rho_1 + \cdots + \rho_{d-1})$.

Under a variation $(\delta\phi_{ij}, \delta\rho_k, \delta X_{ij})$ the matrix $\hat{\sigma} = \dot{\hat{\mathcal{W}}}\hat{\mathcal{W}}^{-1}$ varies as

$$\delta\hat{\sigma} = \dot{\hat{\mathrm{A}}} + [\hat{\mathrm{A}}, \hat{\sigma}] = \sigma^{(\mathrm{r})}_{ij}\hat{\mathrm{r}}_{ij} + \sigma^{(\mathrm{d})}_{k}\hat{\mathrm{d}}_k + \sigma^{(\mathrm{t})}_{ij}\hat{\mathrm{t}}_{ij}, \quad \text{with } \mathrm{A} = \delta\hat{\mathcal{W}} \cdot \hat{\mathcal{W}}^{-1}$$

The measure $\mathcal{D}\hat{\sigma}$ is invariant under the global symmetries $\hat{\mathcal{W}} \to \hat{\mathcal{R}}_0\hat{\mathcal{W}}$ and $\hat{\mathcal{W}} \to \hat{\mathcal{W}}\hat{\mathcal{T}}_0^{-1}$, so that the Jacobian

$$\mathcal{J} = \left|\det\left(\frac{\delta\hat{\sigma}}{\delta(\phi, \rho, X)}\right)\right|$$

does not depend on the $\hat{\mathrm{R}}$ and $\hat{\mathrm{T}}$ variables. Setting then $\hat{\mathrm{R}} = \mathbf{1}$, $\hat{\mathrm{T}} = \mathbf{1}$ *after* having computed the variation $\delta\hat{\sigma}$ we find

$$\begin{aligned}
\sigma^{(\mathrm{d})}_{k} &= \delta\dot{\rho}_k\,, \\
\sigma^{(\mathrm{r})}_{ij} &= \delta\phi_{ij} + (\partial^{-1}\delta\phi_{ij})(\rho_i - \rho_j)\,, \\
\sigma^{(\mathrm{t})}_{ij} &= -e^{\rho_i - \rho_j}\delta\dot{X}_{ij} - 2(\partial^{-1}\delta\phi_{ij})(\dot{\rho}_i - \dot{\rho}_j)\,.
\end{aligned}$$

The Jacobian matrix has thus triangular form and can be computed with the help of the regularization

$$\theta(0) \to \frac{1}{2}, \quad \delta(0) = \frac{1}{h}, \quad \delta'(0) = \frac{1}{h^2}, \quad h \to 0\,. \tag{73}$$

The result (22) is finally obtained making use of the simple identity

$$\sum_{1\le i<j\le d}(\rho_i-\rho_j)=\sum_{j=1}^{d}(d-2j+1)\rho_j=2\sum_{j=1}^{d-1}(d-j)\rho_j\,.$$

Note that the consistency of the regularization prescription (73) can be checked by computing $\langle[\hat{\mathcal{W}}(t)\mathbf{n}_0]^2\rangle$ first directly and that using (22).

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